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UNIFORM TREATMENT OF FLUCTUATIONS AT CRITICAL POINTS

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A generalized critical point is characterized by the vanishing of certain linear relationships. In particular, the dynamics near such a point are non-linear. In this paper, we study fluctuations at such points of spatially homogeneous systems. We discuss thermodynamic critical points as a special case; but the main emphasis is on stochastic kinetic equations. We show that fluctuations at a critical point cannot be characterized by a Gaussian density, but more complicated densities can be used. The theory is applied to the critical harmonic oscillator.

1. Introduction

A generalized critical point can be characterized by the vanishing of certain linear relationships. Such a point may be thermodynamic or kinetic. For example, at the liquid–vapor transition temperature, it is well known that

$$\left.\frac{\partial P}{\partial V}\right|_{T_c} = 0. \tag{1.1}$$

In general, if $\Phi(x)$ is some generalized potential and $\Phi(x_{eq}) = 0$ then the theory of non-equilibrium processes indicates that perturbations from equilibrium evolve according to

$$\dot{x} = -L\frac{\partial \Phi}{\partial x}. \tag{1.2}$$

In the vicinity of equilibrium, x_{eq}, since $\Phi'(x_{eq}) = 0$,

$$\Phi(x) \approx \Phi_{eq} + \frac{\Phi''(x_{eq})(x - x_{eq})^2}{2} + \cdots. \tag{1.3}$$

Defining $1/\chi = \Phi''(x_{eq})$ to be the reciprocal of a "susceptibility" we find

$$\dot{x} = \frac{-L}{\chi}(x - x_{eq}) + \mathcal{O}((x - x_{eq})^2). \tag{1.4}$$

At a generalized critical point, $\Phi'' = 0$. Hence, the dynamics in the vicinity of a critical point are nonlinear.

In this paper, we consider kinetic critical points. As the term is usually used, "critical point" refers to a point in parameter space that is characterized by i) a slowing down of the dynamics and ii) long-range spatial correlations. In this paper, we restrict the problem to spatially homogeneous systems, so that only critical slowing down will be evidenced. A non-equilibrium system is often described by a kinetic equation of the form

$$\dot{x} = b(x, \eta), \tag{1.5}$$

where x is a vector and η is a parameter. Let x_0 be a steady state of (1.5). Let $B(\eta)$ be defined by

$$B(\eta) = (\partial b^i(x, \eta)/\partial x^j)\Big|_{x=x_0}. \tag{1.6}$$

At $\eta = \eta_c$, x_0 is a critical steady state if $B(\eta)$ has a zero eigenvalue. The point $(x_0(\eta_c), \eta_c)$ is a "generalized critical point". In the vicinity of η_c, (1.5) will have nonlinear dynamics. Due to the nonlinear dynamics, the analysis of systems near critical points is quite difficult. Some analysis has been done for one-dimensional systems by Kubo et al.[1]) and Nitzan et al.[2]). In the first paper, two types of critical steady states were defined for one-dimensional systems. We briefly review these. Let $x = 0$ be the steady state. The steady state is of the marginal type[1]) if perturbations from $x = 0$ behave as

$$\dot{x} \sim \pm x^2. \tag{1.7}$$

The origin is a critical type steady state[1]) if perturbations from $x = 0$ behave as

$$\dot{x} \sim \pm x^3. \tag{1.8}$$

Recently, the terminology of Kubo et al. has been generalized to multi-dimensional systems[3]). The paper of Nitzan et al.[2]), in which a one-dimensional system is studied, is contained as a special case of this paper and the accompanying one[4]).

In this paper, we will analyze fluctuations at critical points. Often one reads that fluctuations "become unbounded" or "grow anomalously" at critical points. These statements are meant in the following sense: If one tries to describe fluctuations at a critical point by a Gaussian approximation, then the second moment $\langle x^2 \rangle$ is infinite. We will show that the Gaussian description of fluctuations implicitly assumes linear dynamics. Since critical point dynamics are nonlinear, one should not expect the Gaussian approximation to be valid. Hence, the anomaly is not in the physics, but in the improper use of mathematical approximations. We show that although the Gaussian ap-

proximation is not valid, more complicated densities are appropriate. The technique to demonstrate this will utilize formal asymptotic methods.

Our results are analogous to problems in optics (at a caustic) and wave mechanics (at a classical turning point). In those cases, the geometrical optics and WKB solutions break down, yielding infinite amplitudes. In reality, the intensity of light at a caustic is not infinite, but is large[5]). At a caustic, geometrical optics must be replaced by Airy or Pearcey integrals[5,6]). Similar analyses hold at the classical turning point (e.g.[7,8]).

The main focus of this paper is kinetic critical points, which have a rich dynamical behavior. In section 2, we introduce the stochastic kinetic equation and diffusion approximation. The theory given here is a variation of the mode-mode coupling theory[10]). We discuss a possible resolution of the present controversy regarding a "proper" expansion of the Master Equation to obtain a Fokker–Planck equation[11]). A small parameter arises in the derivation of the diffusion approximation; it characterizes the intensity of fluctuations. The fluctuations are described by a density that satisfies the forward or Fokker–Planck equation. In this paper, techniques for the construction of solutions of the forward equation are given when the underlying deterministic dynamics exhibit critical behavior. In section 3, we derive solutions of the one-dimensional time invariant Fokker–Planck equation. We obtain an exact result, which is then analyzed by asymptotic methods. We obtain a Gaussian density at a non-critical steady state, an Airy density at a marginal type stready state, and a Pearcey density at a critical type steady state. In section 4, these densities are used in a general ansatz ("ray method"[12]) to provide asymptotic solutions of the time dependent multidimensional Fokker–Planck equation. We construct densities in which susceptibilities (i.e. first derivatives) at the critical point are large, but finite. The same result applies to variances. In section 5, we show how our results can be used to construct time-dependent correlation functions. In section 6, we discuss as an example the critical harmonic oscillator[13]) and show how the correlation function is constructed.

Antecedents to this work are found in Kubo et al.[1]), Kitahara[14]), Keizer[15]) and Nitzan et al.[2]). The present work generalizes the results of the above papers.

2. Stochastic kinetic equations, diffusion approximation and Fokker–Planck equation

The thermodynamic theory of critical phenomena cannot be used to treat highly nonequilibrium kinetic phenomena, which are of interest in many areas

of chemistry, physics and biology. Let $\tilde{x}(t)$ denote the statistical variables. Often we can postulate an equation for the mean value of $\tilde{x}(t)$, $x(t)$:

$$\frac{dx^i}{dt} = b^i(x, \alpha), \quad x^i(0) = x_0^i, \quad i = 1, \ldots, n, \quad \alpha = \{\alpha_1, \ldots, \alpha_m\}. \tag{2.1}$$

In order to treat fluctuations, we need to know the kinetic equation that $\tilde{x}(t)$ satisfies. Ideally, one would start with the Liouville equation and derive the kinetic equation. Such a derivation is possible for only the simplest system[21]). Instead, we shall use a generalization of the Langevin method. We will add a zero-mean stochastic term to (2.1). The stochastic function $\tilde{Y}(\tau)$ is characterized by a microscopic time scale, τ, small compared to the macroscopic scale on which measurements are made. Hence

$$\Delta\tau = \eta^2 \Delta t, \tag{2.2}$$

where η is a small parameter. We will not assume that $\tilde{Y}$ has a δ-correlation function and let

$$\gamma^{k\ell} = \int_0^\infty E(\tilde{Y}^k(s)\tilde{Y}^\ell(0))\, ds.$$

We assume that $\tilde{x}(t) = \tilde{x}_\eta(t)$ satisfies the stochastic kinetic equation[22])

$$\frac{dx_\eta^i}{dt} = b^i(\tilde{x}_\eta) + \frac{\sqrt{\epsilon}}{\eta} f_j^i(\tilde{x}_\eta)\tilde{Y}^j(t/\eta^2). \tag{2.3}$$

In eq. (2.3), the convention of summation over repeated indices is used and ϵ is a small parameter characterizing the size of the system and related to the intensity of the fluctuations[1-3,13,15]). Hence $\epsilon \to 0$ corresponds to the thermodynamic limit. The field $f_j^i(x)$ is a given deterministic field. Ideally, one would like to calculate f_j^i from basic principles. Since (2.3) is somewhat ad hoc, a prescription must be given for the calculation of f_j^i. (One such prescription is the fluctuation–dissipation theorem. Another is given by Keizer[22]); also see ref. 3.) As $\eta \to 0$, $x_\eta(t) \to x(t)$, a diffusion process[22]). We set

$$u(x) = E_x\{u_0(\tilde{x}(t)) \mid \tilde{x}(0) = x\}. \tag{2.4}$$

Then $u(x)$ satisfies[22]):

$$\frac{\partial u}{\partial t} = \frac{\epsilon a^{ij}}{2}\frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + c^i \epsilon \frac{\partial u}{\partial x^i} \equiv Lu. \tag{2.5}$$

In (2.5), we use the convention that repeated indices are summed from 1 to n

and have defined:

$$a^{ij} = f^i_k f^j_l (\gamma^{kl} + \gamma^{lk}), \tag{2.6}$$

$$c^i = \gamma^{kl} f^i_k \frac{\partial f^i_l}{\partial x^j}. \tag{2.7}$$

The functions (a^{ij}) form a covariance matrix. In the thermodynamic regime, the fluctuation formalism given for $f^i_j(x)$ should be consistent with the fluctuation–dissipation theorem[15]). In the non-thermodynamic regime, there is more freedom for the choice of fluctuation formalism[3]).

The function $u_0(x)$ plays the role of the initial data when (2.5) is solved. If $\tilde{Y}$ has a δ-correlation function (white noise), then instead of (2.5), one obtains[22]):

$$\frac{\partial u}{\partial t} = \frac{\epsilon a^{ij}}{2} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i}. \tag{2.8}$$

Numerical work[3]) indicates that if the boundaries are non-singular, then eqs. (2.5) and (2.8) yield approximately equivalent solutions for $\epsilon \ll 1$.

We also note that eq. (2.3) is a stochastic equation with correlations and hence is a more general model than a white noise equation. Furthermore, eq. (2.5) [or (2.8)] is derived rigorously – no expansion procedure is needed (compare ref. 11). Equation (2.8) is the backward equation. Usually, in the physical literature the forward or Fokker–Planck equation is used. This equation cannot be derived rigorously; no expansion procedure will rigorously give the Fokker–Planck equation[1,11]). In refs. 24 and 25, it is shown how the Fokker–Planck equation can be obtained from (2.5) or (2.8) by using the theory of partial differential equation. Thus, let

$$v(x, t)\,dx = \Pr\{x \leqslant \tilde{x}(t) \leqslant x + dx\}. \tag{2.9}$$

It can be shown that $v(x, t)$ satisfies, at least weakly, the adjoint (forward) equation

$$v_t = L^* v = \frac{\epsilon}{2}(a^{ij} v)_{ij} - (b^i v)_i - \epsilon (c^i v)_i. \tag{2.10}$$

In the derivation of (2.10), there is a question of boundary terms for u, v as $|x| \to \infty$[25]). For the problems considered here, these questions are relatively unimportant. In (2.10), subscripts indicate differentiation. The approach used here to obtain the Fokker–Planck equation circumvents many of the difficulties of other methods.

The rest of this paper is concerned with obtaining solutions of (2.10) when the underlying deterministic dynamics exhibit critical behavior.

3. One-dimensional problems

In this section, we consider the time independent, one dimensional Fokker–Planck equation

$$\epsilon\frac{(av)_{xx}}{2}-(bv)_x=0 \tag{3.1}$$

subject to

$$\int_{-\infty}^{\infty} v(s)\,ds=1, \quad \lim_{|s|\to\infty} v(s)=0. \tag{3.2}$$

The solution of (3.1, 2) gives the steady state (but not necessarily equilibrium) density for a process satisfying (2.3). In later sections, we generalize the solutions obtained here to solve time dependent, multidimensional problems.

When (3.1) is integrated twice and (3.2) is applied, we find

$$v(x)=k\left[\exp\left\{\int^{x}\frac{2b}{\epsilon a}\,ds\right\}\right], \tag{3.3}$$

where k is the normalization constant

$$k=\int_{-\infty}^{\infty}\exp\left\{\int^{x}\frac{2b}{\epsilon a}ds\right\}dx. \tag{3.4}$$

The main contribution to (3.3) comes from the maximum of the function

$$\Phi(x)=\int^{x}\frac{2b}{a}\,ds. \tag{3.5}$$

We now assume that there is a steady state (i.e. $b(x)=0$ has a solution, x_0). The steady state is classified according to its dynamic behavior.

The *normal type* steady state x_0 is characterized by

$$b(x_0)=0, \quad b'(x_0)\neq 0. \tag{3.6}$$

We are interested in stable steady states, so that it is assumed that $b'(x_0)<0$. Then, perturbations from x_0 decay exponentially. When $\Phi(x)$ is expanded about x_0, we obtain:

$$v(x)\sim k\left[\exp\left\{\frac{-|b'(x_0)|(x-x_0)^2}{\epsilon a}\right\}\right]. \tag{3.7}$$

Thus, we obtain a locally Gaussian density, for small ϵ. This result has also been derived by Kubo et al.[1]) and Keizer[23]) by different arguments. It is also the standard result in the theory of nonequilibrium thermodynamics[18]).

In the marginal case, b depends on one parameter α such that when $\alpha = \alpha_c$ the *marginal type* steady state satisfies

$$b(x_0, \alpha_c) = 0, \quad b'(x_0, \alpha_c) = 0 \quad \text{and} \quad b''(x_0, \alpha_c) \neq 0. \tag{3.8}$$

The canonical dynamics corresponding to the marginal case are[4]):

$$\dot{x} = x^2 - \alpha. \tag{3.9}$$

The flow of such dynamics is sketched in fig. 1. We need to replace the conditions (3.2) by:

$$\lim_{s \to -\infty} v(s) = 0, \quad \int_{-\infty}^{x_F} v(s)\, ds = 1, \tag{3.10}$$

where $x_F < \infty$ is an end value for x [3]).

Since $b'(x_0, \alpha_c) = 0$, the expansion used to obtain (3.7) breaks down. Hence the Gaussian density breaks down. In particular, from (3.7) we have

$$E(x^2) \propto 1/|b'(x_0, \alpha)|^{1/2}. \tag{3.11}$$

Thus, when $\alpha = \alpha_c$, the Gaussian density yields an infinite variance. This is, of course, a purely mathematical divergence and has nothing to do with the physical problem. We take an extra term in the expansion of $\Phi(x, \alpha)$ to obtain

$$v(x) \sim k \exp\left[\frac{b'(x_0, \alpha)(x - x_0)^2}{\epsilon a} + \frac{b''(x_0, \alpha)(x - x_0)^3}{3\epsilon a}\right]. \tag{3.12}$$

A change of variables converts (3.12) to the "Airy density":

$$v(x) \sim \exp\left[-\frac{1}{\epsilon}\{y(x)^3 - \tilde{\alpha} y(x) + \tilde{\beta}\}\right], \tag{3.13}$$

where $y(x)$ is a regular function of x, $\tilde{\alpha}(\alpha)$ is a regular function with $\tilde{\alpha}(\alpha_c) = 0$, and $\tilde{\beta}$ is a constant.

The *critical type* steady state is characterized by two parameters, α, β such that when $\alpha = \alpha_c$ and $\beta = \beta_c$

$$b(x_0, \alpha_c, \beta_c) = b'(x_0, \alpha_c \beta_c) = b''(x_0, \alpha_c, \beta_c) = 0$$
$$\text{and} \quad b'''(x_0, \alpha_c, \beta_c) \neq 0. \tag{3.14}$$

The canonical dynamics of a critical type dynamical system are[4]):

$$\dot{x} = \pm x^3 + \alpha x + \beta. \tag{3.15}$$

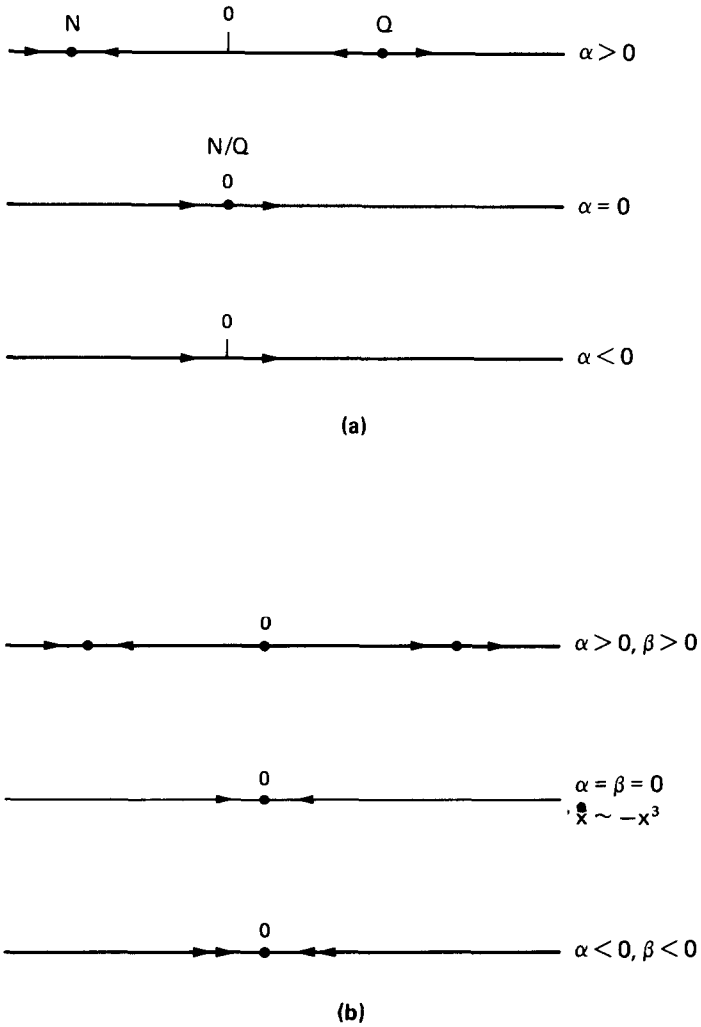

Fig. 1. (a) The marginal type dynamical system has two steady states when a parameter $\alpha > 0$, one degenerate steady state when $\alpha = 0$, and no steady states when $\alpha < 0$. (b) The critical type dynamical system may have three states when $\alpha > 0$, $\beta > 0$, one degenerate steady state when $\alpha = \beta = 0$, and one steady state when $\alpha < 0$, $\beta < 0$.

In this case, the critical values of the parameters are $\alpha_c = \beta_c = 0$. The flow of a critical type system is sketched in fig. 1.

It is clear that the Airy and Gaussian densities both break down when $\alpha = \alpha_c$ and $\beta = \beta_c$. In this case, we take one more term in the Taylor expansion of $\Phi(x)$ and obtain the "Pearcey density":

$$v(x) \sim \exp\left[-\frac{1}{\epsilon}\left\{\frac{y(x)^4}{4} + \frac{\tilde{\alpha}(\alpha,\beta)y(x)^2}{2} + \tilde{\beta}(\alpha,\beta)y(x) + \gamma\right\}\right], \tag{3.16}$$

where $y(x)$, $\tilde{\alpha}(\alpha, \beta)$ and $\tilde{\beta}(\alpha, \beta)$ are regular functions and $\tilde{\alpha}(\alpha_c, \beta_c) = \tilde{\beta}(\alpha_c, \beta_c) = 0$; and γ is a constant.

The results of this section can be obtained by direct use of Levinson's theorem[26]). It is clear that the Gaussian approximation will be valid whenever $|b'(x_0, \alpha)|$ is bounded away from zero.

4. Time dependent, multidimensional Fokker–Planck equation

In this section, we sketch the construction of regular solutions of the time dependent Fokker–Planck equation (2.10). Since Ludwig[5]) has given the construction for the normal case, we only consider marginal and critical type steady states. Our goal is to construct densities that have finite second moments. Exact definitions of marginal and critical type steady states in multidimensional systems are given in the Appendix.

4.1. Marginal type steady state

We seek a solution of (2.10) of the form

$$v(x, t) = \exp\left[\frac{-1}{\epsilon}\left(\frac{\Psi(x, t)^3}{3} - \tilde{\alpha}\Psi(x, t)\right)\right] \sum_{n=0} \epsilon^n z^n(x, t). \tag{4.1}$$

The form of (4.1) is a "ray ansatz"[27]). In it, $\Psi(x, t)$, $\tilde{\alpha}$, and the functions $z^0(x, t)$, $z^1(x, t), \ldots$ are to be determined. In practice we are often interested in just the first term of (4.1). After derivatives are evaluated, terms are collected according to powers of ϵ. We obtain

$$\begin{aligned} 0 = {} & \exp\left[-\frac{1}{\epsilon}\left(\frac{\Psi^3}{3} - \tilde{\alpha}\Psi\right)\right]\left(\frac{1}{\epsilon}\right)\left\{\Psi_t + b^i\Psi_i - (\tilde{\alpha} - \Psi^2)\frac{a^{ij}}{2}\Psi_i\Psi_j\right\} z^0(\tilde{\alpha} - \Psi^2) \\ & + \exp\left[-\frac{1}{\epsilon}\left(\frac{\Psi^3}{3} - \tilde{\alpha}\Psi\right)\right]\left\{b^i_{,i}z^0 + b^i z^0_i - \frac{a^{,ij}_{,i}}{2}(\tilde{\alpha} - \Psi^2)\Psi_i z^0 + z_t\right. \\ & \left. + \frac{a^{ij}}{2}(2\Psi\Psi_i\Psi_j z^0 - (\tilde{\alpha} - \Psi^2)\Psi_{ij}z^0 - 2(\tilde{\alpha} - \Psi^2)\Psi_j z^0_i) - c^i(\tilde{\alpha} - \Psi^2)\Psi_i z^0)\right\} \\ & + \mathcal{O}\left(\epsilon \exp\left[-\frac{1}{\epsilon}\left(\frac{\Psi^3}{3} - \tilde{\alpha}\Psi\right)\right]\right). \end{aligned} \tag{4.2}$$

The leading coefficient of ϵ vanishes if Ψ satisfies

$$\Psi_t + b^i\Psi_i - \frac{(\tilde{\alpha} - \Psi^2)}{2}a^{ij}\Psi_i\Psi_j = 0. \tag{4.3}$$

Equation (4.3) is a generalized eikonal equation[3]). In the rest of this paper, we shall assume that the initial data for v are concentrated at a point so that

$$v(x, 0) = \delta(x - x_0). \tag{4.4}$$

At the deterministic steady states, N, Q (see fig. 2) we set $d\Psi/dt = \Psi_t + b^i\Psi_i = 0$. If Ψ is regular, then $\Psi^2 = \tilde{\alpha}$ at those points. The node N should correspond to a local maximum for $v(x, t)$. Hence we set $\Psi(N) = \sqrt{\tilde{\alpha}}$. Similar reasoning leads to $\Psi(Q) = -\sqrt{\tilde{\alpha}}$.

The value of $\tilde{\alpha}$ is still undetermined. It can be obtained by the following iterative procedure[3]). (If higher order terms are to be considered, then it is necessary to expand $\tilde{\alpha} = \Sigma_{k=0}\, \epsilon^k \tilde{\alpha}_k$. In that case, all of the parameters are determined in a manner analogous to the determination of $\tilde{\alpha}$.) We start at the node N, where $\Psi = \sqrt{\tilde{\alpha}^0}$, the first estimate for $\tilde{\alpha}$. Equation (4.3) can be solved

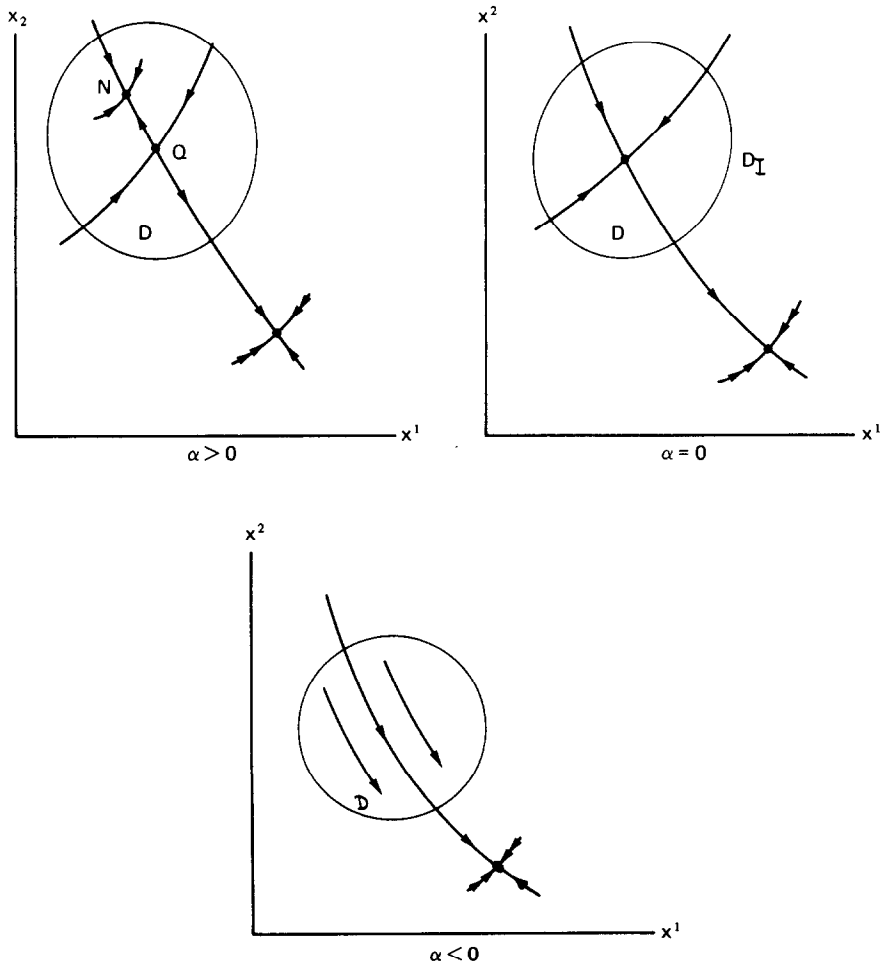

Fig. 2. A two-dimensional marginal dynamical system in a domain D in R^2. When $\alpha < 0$ the deterministic flow is always across D, a problem first studied by Levinson[28]).

by the method of characteristics[12,33]). The characteristic equations are: (where $p_k = \partial\Psi/\partial x^k$ etc.)

$$\frac{dt}{ds} = 1, \tag{4.5a}$$

$$\frac{dx^i}{ds} = b^i - (\tilde{\alpha} - \Psi^2)p_j a^{ij}, \tag{4.5b}$$

$$\frac{d\Psi}{ds} = p_k \frac{dx^k}{ds} + \Psi_t \frac{dt}{ds}, \tag{4.5c}$$

$$\frac{dp_k}{ds} = -\left(2p_k(\Psi a^{ij}p_i p_j) + b^i_{,k}p_i - \frac{(\alpha - \Psi^2)}{2} a^{ij}_{,k}p_i p_j\right). \tag{4.5d}$$

Initial data is given on an ellipsoid surrounding N. As $x \to Q$, the value of $\tilde{\alpha}$ should approach $-\sqrt{\tilde{\alpha}^0}$. If it does not, then a new estimate $\tilde{\alpha}^{(1)}$ is needed. Elsewhere, we have shown that iterates of $\tilde{\alpha}$ can be determined by using the method of false position and that $\tilde{\alpha}$ is a regular function of the deterministic parameter α [3]). When N and Q coalesce (fig. 2b), $\tilde{\alpha} = 0$. After the annihilation of N, Q (fig. 2c), $\tilde{\alpha} < 0$. The stochastic problem for a dynamical system similar to fig. 2c is an old one, studied by Levinson[28]) and Ventcel and Friedlin[29]). Consequently, we restrict ourselves to the dynamic cases represented by figs. 2a,b.

From (4.5b), we see that if $\Psi^2 = \tilde{\alpha}$ on a trajectory, then $dx^i/ds = b^i$, so that the trajectory is a deterministic trajectory. In this way, we will be able to estimate deviations from a given deterministic trajectory. When (4.3) is differentiated with respect to x^k and evaluated on a trajectory we find:

$$\frac{d\Psi_k}{dt} + b^i_{,k}\Psi_k \pm \sqrt{\tilde{\alpha}}\, a^{ij}\Psi_i\Psi_j\Psi_k = 0, \quad k = 1, 2, \ldots n. \tag{4.6}$$

In (4.6) the (+) sign corresponds to trajectories that enter N, the (−) sign to trajectories that enter Q . At either of the steady states, we obtain

$$b^i_{,k}\Psi_k \pm \sqrt{\tilde{\alpha}}\, a^{ij}\Psi_i\Psi_j\Psi_k = 0, \quad k = 1, \ldots n. \tag{4.7}$$

Eq. (4.7) can be solved to yield values of Ψ_k at N or Q.

When N and Q coalesce, so that $\alpha = 0$ and conditions in the Appendix hold, it is possible to show that the Ψ_k can still be calculated. For example, consider the case of only one spatial dimension. Then (4.7) becomes:

$$b_{,x} - \Psi(\mathrm{N})a\Psi_x^2 = 0 \tag{4.8}$$

or

$$\Psi_x^2(\mathrm{N}) = \frac{b_{,x}}{\Psi(\mathrm{N})a}. \tag{4.9}$$

In obtaining (4.8–9), we have replaced $\sqrt{\tilde{\alpha}}$ by $\Psi(\mathrm{N})$. When N, Q coalesce, $b_x \to 0$ and $\Psi(\mathrm{N}) \to 0$. One application of l'Hospital's rule gives

$$\Psi_x^3(\mathrm{N}) = \frac{b_{,xx}(\mathrm{N})}{a(\mathrm{N})}. \tag{4.10}$$

A similar, but more complicated, calculation holds in the multidimensional cases[3]).

The $\mathcal{O}(\exp((-1/\epsilon)(\Psi^3/3 - \alpha\Psi))$ term in (4.2) yields a "transport" equation for z^0 [25,27]). It takes the form[12,25,3]),

$$\frac{dz^0}{ds} + f(s)z^0 = 0, \tag{4.11}$$

where the differentiation is along the characteristic curves (4.5). In this case,

$$z^0(s) = z^0(0) \exp\left[-\int_0^s f(s')\, ds'\right]. \tag{4.11a}$$

When the initial data is concentrated at a point, Ludwig[25]) has shown that the appropriate initial data for z is $z^0(0) = \text{constant}$, chosen for normalization.

4.2. Critical type steady state

For the critical type steady state, instead of (4.1), we seek a solution of (2.10) of the form

$$v(x, t) = \exp\left[-\frac{1}{\epsilon}\left(\tfrac{1}{4}\Psi^4 - \frac{\tilde{\alpha}\Psi^2}{2} - \tilde{\beta}\Psi\right)\right] \sum_{n=0} \epsilon^n z^n(x, t). \tag{4.12}$$

In this case, it is possible to impose the conditions on $v(x, t)$ that

$$v \to 0 \text{ as } \|x\| \to \infty, \quad \int_{-\infty}^{\infty} v(x, t)\, dx = 1 \tag{4.13}$$

and take all of R^n as the domain of interest. Instead of (4.3), we obtain

$$\Psi_t + b^i\Psi_i + \frac{a^{ij}}{2}(\Psi^3 - \tilde{\alpha}\Psi - \tilde{\beta})\Psi_i\Psi_j = 0. \tag{4.14}$$

The value of Ψ at the deterministic nodes $\mathrm{N}_1, \mathrm{N}_2$ and saddle Q (fig. 3) is determined in a manner analogous to the one used in section 4.1. The values of the parameters are also determined in a similar fashion. It is possible to show that all constructions remain regular as the steady states coalesce[3]). The function $z^0(x, t)$ can also be determined in manner analogous to the previous

case, i.e., by solving an ordinary differential equation along the characteristics of (4.14).

5. Correlation functions for critical type systems

In many physical problems, an object of interest is the correlation function

$$R(\tau) = E\{\tilde{x}(t)\,\tilde{x}(t+\tau)\}$$

$$= \int\int x x_1 \Pr\{\tilde{x}(t) \in (x, x+\mathrm{d}x), \tilde{x}(t+\tau) \in (x_1, x_1+\mathrm{d}x_1)\}\,\mathrm{d}x\,\mathrm{d}x_1. \qquad (5.1)$$

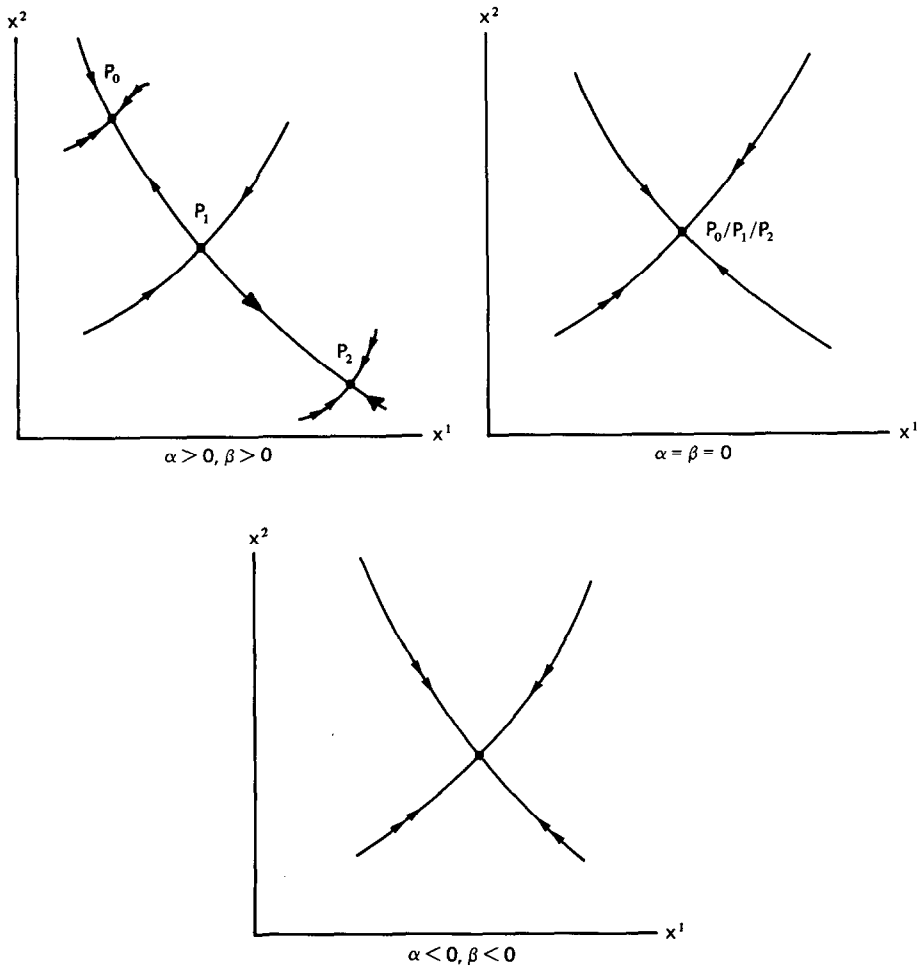

Fig. 3. A two-dimensional critical type dynamical system.

Since the process in our problem is assumed to be stationary, $R(\tau) = E\{\tilde{x}(0)\tilde{x}(\tau)\}$. Now we consider the conditional correlation function:

$$R_{x_0}(\tau) = E\{\tilde{x}(0)\tilde{x}(\tau) \mid \tilde{x}(0) = x_0\}. \tag{5.2}$$

If $v_0(x_0)\,dx_0$ is the initial density for x_0, then,

$$R(\tau) = \int R_{x_0}(\tau) v_0(x_0)\,dx_0. \tag{5.3}$$

However,

$$\begin{aligned} R_{x_0}(\tau) &= \int x_0 x_1 \Pr\{\tilde{x}(\tau) \in (x_1, x_1 + dx_1) \mid \tilde{x}(0) = x_0\}\,dx_1 \\ &= \int x_0 x_1 v_{x_0}(x_1, \tau)\,d_1, \end{aligned} \tag{5.4}$$

where $v_{x_0}(x_1, \tau)\,dx_1$ was calculated in the previous section. Thus

$$R_{x_0}(\tau) = \int x_0 x_1 \exp\left[-\frac{1}{\epsilon}\left(\frac{\Psi(x_1, \tau)^4}{4} - \frac{\tilde{\alpha}\Psi^2}{2} - \tilde{\beta}\Psi\right)\right] z^0(x_1, \tau)\Big|_{x(0)=x_0} dx_1 \tag{5.5}$$

Namely, we start the ray calculation at $x = x_0$ and integrate the ray equations

$$\begin{aligned} \frac{dt}{ds} &= 1, \\ \frac{dx^i}{ds} &= b^i + a^{ij} p_j (\Psi^3 - \tilde{\alpha}\Psi - \tilde{\beta}), \\ \frac{d\Psi}{ds} &= p_k \frac{dx^k}{ds} + \Psi_t \frac{dt}{ds}, \\ \frac{dp_k}{ds} &= -(b^i_{,k} p_i + a^{ij}(3\Psi^2 - \tilde{\alpha}) p_i p_j p_k + \frac{a^{ij}_{,k}}{2}(\Psi^3 - \tilde{\alpha}\Psi - \tilde{\beta}) p_i p_j) \end{aligned} \tag{5.6}$$

until $s = \tau$. Thus (5.5) can be evaluated. The full correlation function, obtained from (5.4) is

$$\begin{aligned} R(\tau) = \int\int x_0 x_1 \exp\left[-\frac{1}{\epsilon}\left(\frac{\Psi(x_1, \tau)^4}{4} - \frac{\tilde{\alpha}\Psi^2}{2} - \tilde{\beta}\Psi\right)\right] z^0(x_1, \tau)\Big|_{x(0)=x_0} \\ \times v_0(x_0)\,dx_1\,dx_0 + \mathcal{O}(\epsilon). \end{aligned} \tag{5.7}$$

In the next section, we give an example of such a calculation.

6. Critical harmonic oscillator

Consider the Langevin equations

$$\dot{x} = v, \tag{6.1}$$

$$\dot{v} = -\alpha x^3 - kx - \gamma v + 2\epsilon\tilde{y}(t). \tag{6.2}$$

These equations describe the Duffing oscillator[30]) subject to random forces[13]). In eq. (6.2), $\tilde{y}(t)$ is assumed to be white noise. Also assume that k in (6.2) depends upon a parameter η in such a manner that $k(\eta_c) = 0$. When $\eta = \eta_c$, one obtains

$$\dot{\alpha} = v, \tag{6.3}$$

$$\dot{v} = -\alpha x^3 - \gamma v + 2\epsilon\tilde{y}(t). \tag{6.4}$$

The origin corresponds to a generalized critical point.

The Fokker–Planck equation for the density $f(t, x, v)$ is

$$\frac{\partial f}{\partial t} = \epsilon \frac{\partial^2 f}{\partial v^2} - v \frac{\partial f}{\partial x} - \frac{\partial}{\partial v}((-\alpha x^3 - kx - \gamma v)f). \tag{6.5}$$

We now seek a solution of (6.5) of the form

$$f(t, x, v) = \exp\left[-\frac{1}{\epsilon}\left(\tfrac{1}{4}\Psi^4 - \frac{\tilde{\alpha}\Psi^2}{2} - \tilde{\beta}\Psi\right)\right] \sum \epsilon^n z^n(x, t), \tag{6.6}$$

where Ψ, $\tilde{\alpha}$, $\tilde{\beta}$ and z^n are to be determined. Following the procedure in section 4, we obtain

$$\Psi_t + v\Psi_x - (k(\eta)x + \alpha x^3 + \gamma v)\Psi_v + (\Psi^3 - \tilde{\alpha}\Psi - \tilde{\beta})\Psi_v^2 = 0. \tag{6.7}$$

Let us now specialize to $\eta = \eta_c$, $k = 0$; i.e. the critical harmonic oscillator. Then $\tilde{\alpha} = \tilde{\beta} = 0$ in (6.6) and (6.7). The ray equations become (with $p_x = \partial\Psi/\partial x$ and $p_v = \partial\Psi/\partial v$) [12,33])

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= v, \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= -\alpha x^3 - \gamma v + 2\gamma p_v \Psi^3, \\
\frac{\mathrm{d}\Psi}{\mathrm{d}t} &= v p_x + \frac{\mathrm{d}v}{\mathrm{d}t} p_v + \Psi_t, \\
\frac{\mathrm{d}p_x}{\mathrm{d}t} &= -6\gamma\Psi^2 p_x p_v + 3\alpha x^2 p_v, \\
\frac{\mathrm{d}p_v}{\mathrm{d}t} &= -6\Psi^2 p_v^3 - p_x + \gamma p_v.
\end{aligned} \tag{6.8}$$

By integrating the ray equations from an initial point $\tilde{x}(t_0) = x_0$, $v(t_0) = v_0$, we obtain the conditional density $f(x, t, v \mid x_0, v_0)$. Then following the procedure in section 5, we can obtain the correlation function.

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Appendix

Marginal and critical type dynamical systems

In this appendix, we give exact conditions for marginal and critical type dynamical systems. This generalizes the scheme of Kubo et al.[1]).

A.1. Marginal type dynamical systems

The deterministic evolution of the macrovariables is governed by

$$\dot{x} = b(x, \eta), \tag{A.1}$$

where $\eta \in \mathrm{R}^1$ is a parameter. Equation (A.1) is assumed to have at most three steady steady states, $Q_0(\eta)$, $Q_1(\eta)$ and P_2. Let B_k be the matrix $(b^i{}_{,j})$ evaluated at Q_0, Q_1 or P_2 ($k = 0, 1, 2$). We assume that:

– For all values of η, B_2 has two real negative eigenvalues. Although P_2 may depend upon η, P_2 is always bounded away from the other steady states.

– As $\eta \downarrow 0$, the distance between $Q_0(\eta)$ and $Q_1(\eta)$ decreases. When $\eta = 0$, Q_0 and Q_1 coalesce and annihilate each other (i.e., when $\eta < 0$, (A.1) has one real and two complex steady states).

– When $\eta > 0$, B_0 has two real negative eigenvalues and B_1 has one real positive and one real negative eigenvalue. When $\eta = 0$, $B_0 = B_1$ has one zero and one real negative eigenvalue. The eigenvector corresponding to the negative eigenvalue has positive slope. The double point $Q_0(0)/Q_1(0)$ is called a saddle node.

A deterministic system satisfying the above assumptions will be structurally similar to the system sketched in fig. 2.

The above conditions can be reformulated by a change of coordinates.

Define the y^1 axis in the direction of the eigenvector of the non-negative eigenvalue of B_1. The y^2 axis is in the direction of the eigenvector of the negative eigenvalue of B_1, with the origin at Q_1. Then

$$\dot{y} = \tilde{b}(y, \eta) \tag{A.2}$$

is the deterministic system in the new coordinates. The system is of the marginal type if:

$$\begin{aligned}
&1)\ \det(\tilde{b}^i{}_{,j}(Q_1, 0)) = 0,\\
&2)\ \tilde{b}^1{}_{,1}(Q_1, 0) = \tilde{b}^2{}_{,1}(Q_1, 0) = 0,\\
&3)\ \tilde{b}^2{}_{,2}(Q_1, 0) \neq 0,\\
&4)\ \tilde{b}^1{}_{,11}(Q_1, 0) - \tilde{b}^2{}_{,11}(Q_1, 0) \neq 0.
\end{aligned} \tag{A.3}$$

The conditions (A.3) have the following interpretation. Condition 1) indicates that the original system has a zero eigenvalue. Condition 2) indicates that when $\eta = 0$ the linear dynamics in the y^1 direction vanish, condition 4) indicates that these dynamics are quadratic. Condition 3) indicates that the second eigenvalue is non-zero.

A.2. Critical type dynamical systems

The macrovariables evolve according to a deterministic kinetic equation

$$\dot{x} = b(x, \eta, \delta), \tag{A.4}$$

where η, δ are one dimensional parameters. The entire bifurcation set of equation (A.4) is still unknown. The physical systems of interest here motivate the following assumptions:

– For some values of η, δ, (A.4) has three steady states $P_0(\eta, \delta)$, $P_1(\eta, \delta)$ and $P_2(\eta, \delta)$. If $B_k = (b^i{}_{,j})$ evaluated at P_k, then when the three steady states are distinct, B_0 and B_2 have real negative eigenvalues. B_1 has one real negative and one real positive eigenvalue. The eigenvector corresponding to the negative eigenvalue has positive slope.

– As η, δ vary, two of the steady states may coalesce and annihilate each other. This behavior is analogous to the marginal bifurcation.

– As η, δ vary, all three steady states may move together and coalesce when $\eta = \delta = 0$. At the critical bifurcation, $B_1 = (b^i{}_{,j})$ has a zero eigenvalue. We assume that the steady state remaining after the critical bifurcation is a stable steady state.

A deterministic system satisfying the above postulates will be structurally similar to the one sketched in fig. 3.

The above properties can be restated in terms of a new coordinate system as follows. The y^1 axis is in the direction of the eigenvector of the non negative eigenvalue of B_1. The y^2 axis is in the direction of the eigenvector of the negative eigenvalue, with the origin at P_1. The deterministic evolution is then

$$\dot{y} = \tilde{b}(y, \eta, \delta). \tag{A.5}$$

A dynamical system is a critical type system if:

$$\begin{aligned}
&1)\ \det(\tilde{b}^i{}_{,j}(P_1, 0, 0)) = 0,\\
&2)\ \tilde{b}^1{}_{,1}(P_1, 0, 0) = \tilde{b}^2{}_{,1}(P_1, 0, 0) = \tilde{b}^1{}_{,11}(P_1, 0, 0) = \tilde{b}^2{}_{,11}(P_1, 0, 0) = 0,\\
&3)\ \tilde{b}^2{}_{,2}(P_1, 0, 0) \neq 0,\\
&4)\ \tilde{b}^1{}_{,111} - \tilde{b}^2{}_{,111} \neq 0.
\end{aligned} \tag{A.6}$$

These conditions have the following interpretation: 1) indicates that the system has a zero eigenvalue, while condition 3) indicates that the second eigenvalue is non-zero. Condition 2) indicates that the linear and quadratic dynamics in the y^1 direction vanish, while 4) indicates that the dynamics are cubic.

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